INTERNATIONAL JOURNAL OF
SCIENCE ARTS AND COMMERCE

ON THE DERTERMINATION OF EXTENSIONS OF CLOSABLE OPERATORS AND THE NUMERICAL RANGE

Justus K. Mile
Kiriri Women's University of Science and Technology
Nairobi.

Abstract

The study of extensions of densely defined operators has led to operators whose image have a real part that is contained wholly in the left hand side of the imaginary line of the complex plane. This is achieved through the fact that when the numerical range of a sesquilinear operator, is not the whole plane, then it is contained in the half plane and is given by

$Re[r\phi(x) + k_0 \|x\|^2] \geq 0 , for\ all\ x \in D(\phi)$. --------------------------- --*(1)*

The discussions led to the function whose image is given be $r\phi(x) + k_0\|x\|^2] \geq 0$ *leading to sesquilinear function whose associated operator is defined by equation* $Re \langle Sx, x\rangle \leq 0\ \ for\ all\ \ x \in D(S),$ *an operator whose real part is less than or equal to zero. This operator is called* $\boldsymbol{dissipative\ operator}$. *In this paper we seek to determine if there can be an extension for our new operator given that its numerical range is a half plane.*

Keywords: Sesquilinear operator, Dissipative operator, closed extension, Half plane.

Definition

Let S be a linear operator. S is called a dissipative operator if

$$Re \langle Sx, x\rangle \leq 0\ \ for\ all\ \ x \in D(S).$$

Our interest is to determine if there can be an extension for our new operator given that its numerical range is a half plane.

Theorem 1

Let B be a Banach space and T be an operator such that $T \in \beta(T)$. Then T is invertible if and only if $\overline{R(T)} = B$. Moreover,

There exists $\alpha > 0$ such that $\|Tx\| \geq \alpha\|x\|$ for all $x \in B$

Theorem 2

Let S be a dissipative operator in H with $D(S)$ dense in H. Then S has a closed dissipative extension $\tilde{S}$ such that

$\sigma(S)$ is contained in the half plane $Re\ \lambda \leq 0$

Proof

S is dissipative, hence

$Re\ \langle Sx, x\rangle \leq 0\ \ for\ all\ \ x \in D(S).$

Multiplying the equation by negative, we have

$$Re\ (-\langle Sx, x\rangle) \geq 0\ for\ all\ \ x \in D(S) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

Let x be such that $\|x\| = 1$, then $\|x\|^2 = 1$.

Adding $\|x\|^2 = 1$ on both sides of inequality (2) we have

$$\|x\|^2 + Re\ (-\langle Sx, x\rangle) \geq \|x\|^2\ \ for\ all\ \ x \in D(S).$$

By definition of a norm, $\|x\|$ is real so is $\|x\|^2$ hence $\|x\|^2 = Re(\|x\|^2)$. Substituting into the above inequality, we get

$$Re\ \|x\|^2 + Re\ (-\langle Sx, x\rangle) \geq \|x\|^2\ \ for\ all\ \ x \in D(S)$$

or

$$Re\ (\|x\|^2 - \langle Sx, x\rangle) \geq \|x\|^2\ \ for\ all\ \ x \in D(S)$$

Since $\|x\|^2 = \langle x, x\rangle = \langle Ix, x\rangle$ we have

$$Re\ (\langle Ix, x\rangle - \langle Sx, x\rangle) \geq \|x\|^2\ \ for\ all\ \ x \in D(S).$$

By definition of inner product, the inequality reduces to

$Re\ (\langle Ix\ - Sx, x\ \rangle) \geq\ \|x\|^2\ \ for\ all\ \ x \in\ D(S),$

then

$Re\ \langle\ (I - S)x, x\rangle \geq\ \|x\|^2.$

This inequality shows that the operator $(I - S)$ is bounded below, hence theorem 1 $(I - S)$ is one to one. Since injectiveness implies the existence of an inverse, $(I - S)$ has an inverse, $(I - S)^{-1}$, defined on $R(I - S)$, that $D(I - S)^{-1}) = R(I - S)$.

Let T be an operator such that

$T = (I + S)(I - S)^{-1}$

Let $x \in D(I - S)^{-1}$, then $x \in D(I - S)$.

Then for $x \in D(I - S)^{-1}$ there is a $y \in R(1 - s)^{-1}$ such that

$y = (I - S)^{-1}x$ ……………………………………………………………………….(3)

For x to be acted upon by $(I - S)^{-1}$ we must have it being acted upon by T too. Thus

$Tx = (I + S)(I - S)^{-1}x$ for all $x \in D(I - S)^{-1}$ ……………………………………….(4)

This implies that the $D(T)\ =\ D(I - S)^{-1}) = R(I - S)$.

Thus

$D(T) = R(I - S).$

Substituting y with $(I - S)^{-1}x$ in equation $|\phi(x_n)| \leq\ 2\,\beta\ \phi_1(x_n)^{\frac{1}{2}}\,\phi_1(x_n - x_m)^{\frac{1}{2}} + \|Tx_n\|\|x_m\|$. ……………………………………………(5)

we have

$Tx = (I + S)y$ ……………………………………………………………………………………(6)

This implies that y is in the domain of $(I+S)$ which is also in the range of $(I-S)^{-1}x\ for\ all\ y \in (I-S)$. Since this is true for all $x \in\ D(I-S)^{-1}$ all $y \in R(I-S)^{-1}$, we have that

$$D(I+S)\ = R(1-S)^{-1}.$$

We now determine if T is bounded. From equation (6), $Tx = (I+S)y$, we have

$$\|Tx\| = \|(Iy+Sy)\| = \|(y+Sy)\|$$

$$\|Tx\|^2 = \|(y+Sy)\|^2$$

By the definition of a norm induced by inner product, we have

$$\|Tx\|^2 = \|(y+Sy)\|^2 = \langle y+Sy, y+Sy\rangle.$$

On expansion, we get

$$\langle\, y+Sy, y+Sy\,\rangle = \langle\, y,y\,\rangle\ +\langle\, y,Sy\,\rangle\ +\langle\, Sy,y\,\rangle\ +\langle\, Sy,Sy\,\rangle.$$

But $\langle\, y,y\,\rangle\ =\ \|y\|^2$ Thus on simplification, we have

$$\langle\, y,y\,\rangle\ +\langle\, y,Sy\,\rangle\ +\langle\, Sy,y\,\rangle\ +\langle\, Sy,Sy\,\rangle\ = \|y\|^2\ +\langle\, y,Sy\,\rangle\ +\langle\, Sy,y\,\rangle\ +\ \|Sy\|^2.$$

Since the sum of two conjugates is twice their real part, we have

$$\|y\|^2\ +\langle\, y,Sy\,\rangle + \langle\, Sy,y\,\rangle +\ \|Sy\|^2 =$$
$$\|y\|^2\ +\ 2\,Re\langle\, Sy,y\,\rangle +\ \|Sy\|^2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (7)$$

Since S is dissipative , $Re\langle\, Sy,y\,\rangle \le\ 0$ consequently, $2Re\langle Sy,y\,\rangle \le\ 0.$

Thus by multiplying the inequality by negative one, we have $-2Re\langle\, Sy,y\,\rangle \ge 0$.

Substituting the inequality in right hand side of equation (7) we have

$$\|y\|^2\ +\ 2\,Re\langle\, Sy,y\,\rangle + \|Sy\|^2 \le\ \|y\|^2\ -\ 2\,Re\langle\, Sy,y\,\rangle + \|Sy\|^2$$

On expanding the middle term, we have

$$= \|y\|^2\ -\langle\, y,Sy\,\rangle\ -\langle\, Sy,y\,\rangle +\ \|Sy\|^2$$

$$= \langle\, y,y\,\rangle\ -\langle\, y,Sy\,\rangle\ -\langle\, Sy,y\,\rangle + \langle\, Sy,Sy\,\rangle$$

$$= \langle\, y - Sy, y - Sy \,\rangle$$

$$= \|y - Sy\|^2$$

$$= \|(I - S)y\,\|^2$$

Finally we have

$$\|Tx\|^2 \le \|(I - S)y\,\|^2.$$

From equation (3.14), $y = (I - S)^{-1}\, x$, hence $x = (I - S)y$ and $\|x\| = \|(I - S)y\|$ and $\|x\|^2 = \|(I - S)y\|^2$.

Thus

$$\|Tx\|^2 \le \|(I - S)y\|^2 = \|x\|^2.$$

Hence

$$\|Tx\|^2 \le \|x\|^2$$

and

$$\|Tx\| \le \|x\|$$

$$\|Tx\| \le \|x\| \;\; for\ all\ \ x \in D(T) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots. (8)$$

We now find the extension of T and determine if it is bounded.

Thus, extend T to $\overline{D(T)}$. That is, if $x \in \overline{D(T)}$ then it is an accumulation point. That is there is a sequence $\{x_n\}$ of elements of $D(T)$ such that $x_n \to x$.

By (8), Tx_n is a Cauchy sequence in , since such sequences are convergent, it has a limit say z.

Define to be $\overline{T}x = z$. We now check that this definition is independent of the sequence chosen and that $\overline{T}$ is an extension of T to $\overline{D(T)}$.

Let x be any element of H, then by Projection theorem $x = w + y$, where $w \in \overline{D(T)}$ and y is orthogonal to $D(T)$.

Define $\tilde{T}x = \overline{T}w$

then

$$\|\tilde{T}x\| = \|\overline{T}w\| \leq \|w\| \leq \|x\|.$$

Thus $\tilde{T} \in B(H)$ and since $\|\tilde{T}x\| \leq \|x\|$ implies

$$\|\tilde{T}\| \leq 1 \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots (9)$$

Now by (4) and (6)

$$x = y - Sy, Tx = y + Sy \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots (10)$$

Adding these two we have $2y = x + Tx$ and subtracting the two, we have $2Sy = Tx - x$.

Thus, we have the system

$$2y = x + Tx, 2Sy = Tx - x \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots (11)$$

From $2y = x + Tx$, we have by linearity if $(I + T)$ that $2y = (I + T)x$, we therefore substitute for y using equation (3.14),$y = (I - S)^{-1}x$ to get

$$2(I - S)^{-1}x = (I + T)x, \text{ thus}$$

$$2(I - S)^{-1} = (I + T) \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots (12)$$

implies that $(I + T)$ has an inverse, thus invertible, consequently, one to one operator.

Also, from the relation above, we get that

$$R(I + T) = R((1 - S)^{-1}) = D(1 - S) = D(S).$$

Therefore, $R(I + T) = D(S)$.

Now $D((I + T)^{-1}) = R(I + T) = D(S)$.

Likewise, from equation (11), $2Sy = Tx - x$ can be written as $2Sy = (T - I)x$. Substituting for y using $y = (I - S)^{-1}x$ to get

$$2S(I - S)^{-1}x = (T - I)x, \text{ thus}$$

$$2S(I-S)^{-1} = (T-I) \quad \ldots\ldots (13)$$

Dividing equation (13) by (12), we get

$$\frac{2S(I-S)^{-1}}{2(I-S)^{-1}} = \frac{T-I}{I+T}.$$

Hence

$$S = \frac{T-I}{I+T}$$

$$S = (T-I)(I+T)^{-1} \quad \ldots\ldots (14)$$

A candidate for the extension $\tilde{S}$ is

$$\tilde{S} = (\tilde{T}-I)(I+\tilde{T})^{-1} \quad \ldots\ldots (15)$$

with $D(\tilde{S}) = R(I+\tilde{T})$.

Just like in equation (8) where we had to validate it after confirming that $(I+T)$ is one to one hence is invertible, we cannot assume that (15) holds without determining the inevitability of $(I+\tilde{T})$.

Let $x \in H$ such that

$$(I+\tilde{T})x = \overline{0} \quad \ldots\ldots (16)$$

Let y be any element of H;

set $z = (I+\tilde{T})y, then\ z = y + \tilde{T}y$

and $z - y = \tilde{T}y$.

Applying the norm function, we get $\|z-y\| = \|\tilde{T}y\|$, By triangle inequality for norm, we have

$\|z-y\| = \|\tilde{T}y\| \leq \|\tilde{T}\|\|y\|$.--(17)

Due to boundedness of $\tilde{T}$ in (17) we have

$$\|z-y\| = \|\tilde{T}y\| \leq \|\tilde{T}\|\|y\| \leq \|y\|$$

Hence $\|z-y\| \leq \|y\|$

For some α a positive real number $\|z-y+\alpha x\| \leq \|y-\alpha x\|$.

Factorizing -1 on the left hand side and squaring both sides, we get

$\|z-y+\alpha x\|^2 \leq \|y-\alpha x\|^2$

$\|z-(y-\alpha x)\|^2 \leq \|y-\alpha x\|^2$

We the expand the inequality on both sides by using definition of the norm generated by an inner product to get

$\langle z-(y-\alpha x), z-(y-\alpha x)\rangle \leq \|y-\alpha x\|^2$.

On expansion, we get

$$\langle z,z\rangle - \langle z, y-\alpha x\rangle - \langle y-\alpha x, z\rangle + \langle y-\alpha x, y-\alpha x\rangle \leq \|y-\alpha x\|^2.$$

Which reduces to

$$\|z\|^2 - 2Re\langle z, y-\alpha x\rangle + \|y-\alpha x\|^2 \leq \|y-\alpha x\|^2$$

$$\|z\|^2 - 2Re\langle z, y-\alpha x\rangle \leq 0$$

$$\|z\|^2 - 2Re\langle z, y\rangle + 2Re\langle z, \alpha x\rangle \leq 0$$

$$\|z\|^2 - 2Re\langle z, y\rangle + 2\alpha Re\langle z, x\rangle \leq 0.$$

Dividing through by $\propto$ gives

$$\frac{\|z\|^2}{\alpha} - \frac{2Re\langle z, y\rangle}{\alpha} + 2\,Re\langle z, \quad x\,\rangle \leq 0$$

letting $\alpha \to \infty$ this gives

$2\,Re\,\langle\, z,\ x\,\rangle\ \leq\ 0$ which reduces to

$Re\,\langle z, x\,\rangle \leq 0 \qquad z \in R(I + \tilde{T})$.-- (18)

Since $R(I + \tilde{T}) \supseteq R(I + T) = D(T)$

We see that $R(I + T)$ is dense in H

Hence there is a sequence $\{z_n\}$ of elements in $R(I + \tilde{T})$ such that $z_n \to x \in H$

Thus (18) implies $Re\,\|x\|^2 \leq 0$ which shows that $x = \overline{0}$

Thus the operator $\tilde{0}$ given by (14) is well-defined and it is clearly an extension of S.

We show that $\tilde{S}$ is closed.

Suppose $\{x_n\}$ is a sequence of elements $D(\tilde{S}) = R(I + \tilde{T})$ such that

$$x_n = (I + \tilde{T})w_n$$

and

$$w_n = (I + \tilde{T})^{-1}x_n.$$

By equation (14)

$$\tilde{S}x_n = (\tilde{T} - I)w_n\,.$$

Adding $x_n = w_n + \tilde{T}w_n \ \ to\ -Sx_n = \tilde{T}w_n - w_n\ \ yields\ \ x_n - Sx_n = 2w_n$.

Hence

$$2w_n = x_n - \tilde{S}x_n \to x - h \quad as\ \ n \to \infty$$

which is equivalent to

$$2w_n = (I - \tilde{S})x_n \to x - h \ as \ n \to \infty.$$

Without loss of generality $w_n \to x - h \ as \ n \to \infty$

From the above equation we have $w_n = (I + \tilde{T})^{-1} x_n$ we manipulate it to be $x\_n = (I + \tilde{T})w_n$. Multiplying $x_n = (I + \tilde{T}) \ w_n$and $\widetilde{S}\, x_n = (\tilde{T} - I)w_n$ by 2, we get $2x_n = 2(I + \tilde{T})w_n$and $2\tilde{T}x_n = 2(\tilde{T} - I)w_n$

Since $\tilde{T} \in B(H)$, this implies

$$2x_n = 2(I + \tilde{T})w_n \to (I + \tilde{T})(x - h)$$

$$2\, \tilde{S}x_n = 2(\tilde{T} - I)w_n \to (\tilde{T} - I)(x - h)$$

from which we conclude

$$2x = (I + \tilde{T})(x - h), 2h = (\tilde{T} - I)(x - h) \ since \ x_n \to \ x \ and \ \tilde{S}x_n \to \ h$$

In particular, we see that

$$x \in R(I + \tilde{T}) = D(\tilde{S}),$$

and

$$\tilde{S}x = (\tilde{T} - I)(I + \tilde{T})^{-1}x = \tfrac{1}{2}(\tilde{T} - I)(x - h) = h$$

hence $\tilde{S}$ is a closed operator.

We also show that $\widetilde{S}$ is a dissipative operator.

For $w = (I + \tilde{T})^{-1}x$, $x = (I + \tilde{T})w$

$$\tilde{S}x = (\tilde{T} - I)(I + \tilde{T})^{-1}x = (\tilde{T} - I)w.$$

Therefore,

$$\langle \tilde{S}x, x \rangle = \{\langle (\tilde{T} - I)w, (I + \tilde{T})w \rangle\}$$

$$= \|\tilde{T}w\|^2 - \langle w, \tilde{T}w \rangle + \langle \tilde{T}w, w \rangle - \|w\|^2.$$

Since a norm is always real and range of an inner product is in a complex field (because we are working with complex inner product space)

$$Re\langle \tilde{S}x, x \rangle = \|\tilde{T}w\|^2 - \|w\|^2 \leq \|\tilde{T}\|^2 \|w\|^2 - \|w\|^2 = (\|\tilde{T}\|^2 - 1)\|w\|^2.$$

By (3)

$$Re\langle \tilde{S}x, x \rangle = \|\tilde{T}w\|^2 - \|w\|^2 \leq 0 \quad \ldots\ldots (19)$$

Finally, we must verify that $\lambda \in \rho(\tilde{S})$ for $Re\,\lambda > 0$. That is, the resolvent set is the positive half plane excluding the vertical imaginary axis. If that is the case, then $\lambda \in \sigma(\tilde{S})$ will be contained in the negative half plane including the vertical axis. This is because the two sets are disjoint and one is the complement of the other and also due to the fact that $\lambda \in \sigma(\tilde{S})$ is closed and $\lambda \in \rho(\tilde{S})$ is open.

$$Re\,\{\langle (\tilde{S} - \lambda I)x, x \rangle\} = Re\,\langle (\tilde{S}x - \lambda x, x \rangle = Re\,\langle \tilde{S}x, x \rangle - Re\langle \lambda x, x \rangle = Re\,\langle \tilde{S}x, x \rangle - Re(\lambda\langle x, x \rangle)$$

$$= Re\langle \tilde{S}x, x \rangle - Re\,\lambda\|x\|^2.$$

Since $\tilde{S}$ is dissipative we have

$$Re\,\{\langle (\tilde{S} - \lambda I)x, x \rangle\} \leq -(Re\,\lambda)\|x\|^2.$$

Multiplying through by -1, we get

$$Re\,\lambda\|x\|^2 \leq -Re\,\{\langle (\tilde{S} - \lambda I)x, x \rangle\} \leq (\tilde{S} - \lambda I)x\|x\|.$$

Hence $\|(\tilde{S} - \lambda I)x\|\|x\| \geq Re\lambda\|x\|^2$.

This implies that $(\tilde{S} - \lambda I)$ is bounded below hence one to one for $Re\,\lambda > 0$.

Thus all we need to show is that $R(\tilde{S} - \lambda I) = H \; for \; Re\lambda > 0$.

Now

$$(\tilde{S} - \lambda I) = [(I - \lambda)\tilde{T} - (I + \lambda)](I + \tilde{T})^{-1}.$$

It is possible to solve

$$(\tilde{S} - \lambda I)x = z \quad \ldots\ldots (20)$$

If and only if one can solve $[(I-\lambda)\tilde{T} \; - (I+\lambda)]\left(I+\tilde{T}\right)^{-1}x$ which is equivalent to

$$[(I-\lambda)\tilde{T}-(I+\lambda)]w = z \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (21)$$

where $w = \left(I+\tilde{T}\right)^{-1}x$.

Equation (21) can be solved for all $z \in H$ when $Re\,\lambda > \; 0$.

Dividing (21) $by\, I-\lambda$ we get

$$\left[\tilde{T}-\frac{I+\lambda}{I-\lambda}\right]w = \frac{z}{1-\lambda}$$

This is obvious for all $\lambda = \; 1$

If $\lambda \neq \; 1$, all we need to note is that for $Re\,\lambda > \; 0$,

$$\left|\frac{I+\lambda}{I-\lambda}\right| > \; 1$$

.

Since $\|\tilde{T}\|, is\ bounded\ , that\ is, \|\tilde{T}\| \leq \; 1,\ \frac{I+\lambda}{I-\lambda}\ is\ in\ \rho(\tilde{T})$

hence (21) can be solved for all $z \in \; H$. □

If $\overline{W(T)}$ is a half-plane then Theorem 2 gives a closed extension $\tilde{T}$ of T satisfying

$$\sigma\left(\tilde{T}\right) \subseteq \overline{W(T)} \subseteq \overline{W(\tilde{T})}.$$

In fact, all we need to do is to define S by 1 for appropriate r, k then extend S to $\tilde{S}$ by as

$S = rT - kI.$

The extension $\tilde{T}$ define by $\tilde{T} = \frac{\tilde{S} + kI}{r}$ has all the desired properties. Hence theorem 2 implies the following results.

Theorem 3

Let T be a densely defined linear operator on H such that $\overline{W(T)}$ is a half - plane. Then T has a closed extension $\tilde{T}$ satisfying

$$\sigma(\tilde{T}) \subseteq \overline{W(T)} \subseteq \overline{W(\tilde{T})}.$$

Proof

If T be a densely defined linear operator on H such that $\overline{W(T)}$ is a half - plane, by theorem 1, T has a closed extension such that the resolvent set is in a half plane which is a complement of the numerical range.

This implies that the spectrum of this operator is in the other half plane, $Re\,\lambda \leq 0$ hence contained in the $\overline{W(T)} \subseteq \overline{W(\tilde{T})}$.

References

1. Martin, S. (2009). Principles of Functional Analysis, (2nd ed.).
2. Bachman, G. & Narici, L. (2000). Functional Analysis, New York and London: Academics Press.
3. Bonsall, F. & Duncan, J. (1980). Studies in Functional Analysis. American Mathematical Society, 21, 1-9.
4. Halmos, P.R. (1967). Hilbert Space Problem Book. Princetone: D. Von Nostrand.
5. Philips, R.S. (1959). Dissipative Operators and Parabolic Partial Differential Equations. Communication on Pure and Applied Mathematics. 12.249-276.
6. Kato, T. (1961). Fractional Powers of Dissipative Operators. Journal of the Mathematical Society of Japan .13, 246-274.$https:// projecteuclid.org/download/pdf_1/Euclid.jimsi/1261062709