INTERNATIONAL JOURNAL OF
SCIENCE ARTS AND COMMERCE

SPECTRAL PROPERTIES OF COMMUTANTS OF UNBOUNDED SELF-ADJOINT OPERATORS WITH SIMPLE SPECTRA

Fidelis Musena Mukudi[1], Prof. Justus Kitheka Mile[2], & Dr. Lucy Chikamai[2], Prof. Shem Omukunda Aywa[2]

Fidelis Musena Mukudi

Kibabii University

Prof. Justus Kitheka Mile

Kiriri Women's University of Science and Technology

Dr. Lucy Chikamai

Kibabii University

Prof. Shem Omukunda Aywa

Kibabii University

ABSTRACT

An operator B is a commutant of the unbounded Self-adjoint operator with simple spectra F if $BF \subseteq FB$. The properties of the commutant are determined by those of the operator F. In this article, we show that the spectrum of these commutants is a subset of the real number set. We also establish the effect of the spectral properties of the unbounded Self-adjoint operators with simple spectra to the spectrum of its commutant. Finally, we show that the spectral measure of the unbounded Self-adjoint operator with simple spectra is a scalar multiple of that of its commutant.

Keywords: Unbounded operators, Self-adjoint operators, spectral theorem, operators with simple spectra

1. INTRODUCTION

The concept of commutativity is of great importance, especially in quantum mechanics. For instance, it enables one to measure the quantity of the two observables, simultaneously, without the need of invoking the uncertainty relation, [3]. Our interest is to look at the commutativity relation with respect to the unbounded operators, specifically, the commutants of the unbounded Self-adjoint operators with simple spectra. An operator B is a commutant of the unbounded operator F if $BF \subseteq FB$. The spectrum of Self-adjoint operators is known to be a subset of the real number set, [1]. This fact is true for both bounded and unbounded operators. However, the spectrum of their commutants in relation to those of these operators, specifically, the unbounded Self-adjoint operators with simple spectra has not been yet established. This paper seeks to establish these relations and in turn, characterize the spectral properties of the commutants of unbounded Self-adjoint operators with simple spectra.

The commutants of unbounded Self-adjoint operators are said to be bounded, [1]. The boundedness is a tool in helping us explore its properties. From the commutativity relation of two operators, the properties of one operator most likely to determine those of the other. In particular, the spectral properties of the unbounded Self-adjoint operator with simple spectra most likely determines those of its commutants. To establish to what extent this is true, we will use the spectral theorem of unbounded Self-adjoint operators as well as the properties of these operators. Our underlying space will be a complex Hilbert space.

2. PRELIMINARY CONCEPTS

In this section, we discuss, in brief, some basic concepts required to effectively discuss the results of our paper. We will use $\mathbb{H}$ to denote a Complex valued Hilbert space over a field $\mathbb{F}$. Our unbounded Self-adjoint operator whose properties will be exploited will be denoted by F while its spectral measure by $P(\lambda)$ for $\lambda \in \sigma(F)$ where $\sigma(F)$ is the spectrum of the operator F. The restriction of an operator F on $\mathbb{H}_i \subseteq \mathbb{H}$ will be denoted by $F|_{\mathbb{H}_I}$. We begin by providing a few definitions for the terms that will commonly be used in this paper.

Definition: Unbounded operator

Let $F : D(F) \subseteq \mathbb{H}_1 \rightarrow \mathbb{H}_2$ be an operator, then F is said to be unbounded. Furthermore, there is no positive constant M such that $\|Fx\| \leq M\|x\|$. The operator F is densely defined if $\overline{D(F)} = \mathbb{H}_1$.

Definition: Commutant of an unbounded operator

An operator B is a commutant of the unbounded operator F if $BF \subseteq FB$.

Definition: Operator with simple spectra

An operator F has a simple spectra if the multiplicity of any $\lambda \in \sigma(F)$ is 1.

To understand the spectral properties of the unbounded Self-adjoint operators, we need to understand what a spectral measure is. In the same spirit, we provide the definition.

Definition: Spectral measure

Let X be a set and $\mathfrak{x}$ its σ-algebra, then the operator $P(\cdot)$ from $\mathfrak{x}$ to the Hilbert space $\mathbb{H}$ is a spectral measure if

i. $P(\theta)$ is an orthogonal projection, that is, $P^2(\theta) = P(\theta)$ and $P^*(\theta) = P(\theta), \theta \in \mathfrak{x}$
ii. $P(X) = 1$
iii. $P(\bigcup_{i=1}^{\infty} \theta_i) = \sum_{i=1}^{\infty} P(\theta_i)$ for $\theta_i \in \mathfrak{x}, i \in \Lambda$ such that $\theta_i \cap \theta_j = \emptyset$ for $i \neq j$ and $\bigcup_{i=1}^{\infty} \theta_i = X$.

The structure of Self-adjoint operators makes it easier to study its properties as well as understand the nature of spaces onto which it acts upon. Its structure is summarized by the spectral theorem which is provided for bounded as well as the unbounded version. The spectral decomposition of the unbounded Self-adjoint operator states that if F is an unbounded Self-adjoint operator on a Hilbert space $\mathbb{H}$, then there exists a unique spectral measure P_F, dependent on F, on the Borel sigma-algebra $B(\mathbb{R})$ such that

$$F = \int_{\mathbb{R}} \lambda \, dP_F(\lambda), \qquad [5]$$

the unbounded Self-adjoint operators can be found in [1, 2, 5]. In [5], we get that the commutants of unbounded Self-adjoint operators, F also commutes with the spectral measures of the operators,

$P_F(\lambda)$. For convenience, we use $P(\lambda)$ to imply $P_F(\lambda)$ when there is no room for confusion. Therefore, we have the relation, if $BF \subseteq FB$, then $BP_F(\lambda) = P_F(\lambda)B$, to be precise, $BP(\lambda) = P(\lambda)B$.

We now proceed to our results.

3. SPECTRAL PROPERTIES OF THE COMMUTANTS

This section provides the results of this article. We provide the expression for the numerical range of the commutants of the unbounded self-adjoint operators with simple spectra. The expression is fundamental in characterizing the spectral properties of the commutants.

Proposition 3.1

The numerical range of the commutants of the unbounded Self-adjoint operators with simple spectra is real.

Proof

Let F be the unbounded operator with simple spectra and B its bounded commutant. Then, $BF \subseteq FB$.

Since F is unbounded Self-adjoint operator, by the spectral theorem, [5] there exists a unique spectral measure P_F, dependent on F, on the Borel sigma-algebra $B(\mathbb{R})$ such that

$$F = \int_{\mathbb{R}} \lambda \, dP_F(\lambda), \quad \lambda \in \mathbb{R}.$$

From the relation, $BF \subseteq FB$, we have $BP_F(\lambda) = P_F(\lambda)B$ or $BP(\lambda) = P(\lambda)B$ [2].

The underlaying space for the unbounded Self-adjoint operators with simple spectra is a separable Hilbert space. Further, the orthogonal direct sum of the null spaces of $\mathbb{H}$ gives us the whole space, $\mathbb{H}$. Thus

(3.1) $$\mathbb{H} = \bigoplus_{i \in \mathbb{N}} \mathbb{H}_i$$

where the null spaces of $F - \lambda I$ are $\mathbb{H}_i$ [4]. These null spaces are the span of the eigenvectors of the operator F. The unbounded Self-adjoint operators with simple spectra have cyclic vectors and each of the null spaces $\mathbb{H}_i$, has at least one cyclic vector, [5]. Let x be a cyclic vector of the operator $F|_{\mathbb{H}_I}$ such that $\|x\| = 1$. If $\lambda_i \in \sigma(F|_{\mathbb{H}_I})$, then

(3.2) $$\text{Span}(P(\lambda_i)x) = \mathbb{H}_i \quad [5].$$

Let $x \in D(P(\lambda_i))$, then

$$\langle BP(\lambda_i)x, x \rangle = \langle P(\lambda_i)x, B^*x \rangle$$
$$= \langle P^2(\lambda_i)x, B^*x \rangle$$

$= \langle P(\lambda_i)x, P^*(\lambda_i)B^*x\rangle$

$= \langle P(\lambda_i)x, P(\lambda_i)B^*x\rangle$

Thus, $B^*x \in D(P(\lambda_i)$ for $\lambda_i \in \sigma\left(F|_{\mathbb{H}_I}\right)$. Since $\text{Span}(P(\lambda_i)x) = \mathbb{H}_I$, there exists a nonzero $k \in \mathbb{F}$ such that $B^*x = k\,P(\lambda_i)x.$

Therefore

$\langle B^*x, x\rangle = \langle kP(\lambda_i)x, x\rangle$

$\langle x, Bx\rangle = \langle kP^2(\lambda_i)x, x\rangle$

$= k\langle P^2(\lambda_i)x, x\rangle$

$= k\langle P(\lambda_i)x, P(\lambda_i)x\rangle$

$= k\|P(\lambda_i)x\|^2$

We have $\langle x, Bx\rangle = k\|P(\lambda_i)x\|^2$, taking conjugates and simplifying further, we have

$\langle x, Bx\rangle = k\|P(\lambda_i)x\|^2$

$\overline{\langle x, Bx\rangle} = \overline{k\|P(\lambda_i)x\|^2}$

$\langle Bx, x\rangle = \overline{k}\ \overline{\|P(\lambda_i)x\|^2}$

$\langle Bx, x\rangle = \overline{k}\ \|P(\lambda_i)x\|^2$

The relation

(3.3) $$\langle Bx, x\rangle = k\,\|P(\lambda_i)x\|^2$$

is true only if $k \in \mathbb{R}$. Therefore, $\langle Bx, x\rangle = k\|P(\lambda_i)x\|^2 \in \mathbb{R}$, if $k \in \mathbb{R}$. Thus, $\langle Bx, x\rangle \in \mathbb{R}$.

We now prove that $k \in \mathbb{R}$.

Let $k = a + b\mathrm{i}$ for $a, b \in \mathbb{R}$ then $\langle Bx, x\rangle = k\|P(\lambda_i)x\|^2$ is equivalent $\langle Bx, x\rangle = (a + b\mathrm{i})\|P(\lambda_i)x\|^2$. Let $\alpha_i \in \sigma\left(B|_{\mathbb{H}_i}\right)$ and y the corresponding eigenvector. Since $y \in \mathbb{H}_i$ and $\mathbb{H}_i$ is one dimensional because F has a simple spectra, there is a nonzero $c \in \mathbb{F}$ such that

(3.4) $$y = cx \quad or \quad x = \frac{1}{c}y$$

Then

$$\langle Bx, x\rangle = (a + b\mathrm{i})\|P(\lambda_i)x\|^2$$

$$\langle Bx, x\rangle = a\|P(\lambda_i)x\|^2 + b\mathrm{i}\|P(\lambda_i)x\|^2$$

$$\left\langle B\left(\frac{1}{c}y\right), \left(\frac{1}{c}y\right)\right\rangle = a\left\|P(\lambda_i)\left(\frac{1}{c}y\right)\right\|^2 + b\mathrm{i}\left\|P(\lambda_i)\left(\frac{1}{c}y\right)\right\|^2$$

$$\frac{1}{|c|^2}\langle By, y\rangle = \frac{a}{|c|^2}\|P(\lambda_i)y\|^2 + \frac{b}{|c|^2}\mathrm{i}\|P(\lambda_i)y\|^2$$

$$\langle By, y\rangle = a\| P(\lambda_i)y\|^2 + b\mathrm{i}\| P(\lambda_i)y\|^2$$

$$\overline{\alpha}_i\langle By, y\rangle = \overline{\alpha}_i a\| P(\lambda_i)y\|^2 + \overline{\alpha}_i b\mathrm{i}\| P(\lambda_i)y\|^2$$

$$\langle By, \alpha_i y\rangle = \overline{\alpha}_i a\| P(\lambda_i)y\|^2 + \overline{\alpha}_i b\mathrm{i}\| P(\lambda_i)y\|^2$$

$$\langle By, By\rangle = \overline{\alpha}_i a\| P(\lambda_i)y\|^2 + \overline{\alpha}_i b\mathrm{i}\| P(\lambda_i)y\|^2$$

$$\| By\|^2 = \overline{\alpha}_i a\| P(\lambda_i)y\|^2 + \overline{\alpha}_i b\mathrm{i}\| P(\lambda_i)y\|^2$$

The constant$\| By\|^2 \in \mathbb{R}$, as such $\| By\|^2 = \overline{\alpha}_i a\| P(\lambda_i)y\|^2 + \overline{\alpha}_i b\mathrm{i}\| P(\lambda_i)y\|^2$ only if the right-hand side is real. This means that $\alpha_i \in \mathbb{R}$ and $b = 0$. Consequently, $k = a$ as required.

Finally, we show that $\langle Bx, x\rangle \in \mathbb{R}$ for all elements in $\mathbb{H}$. Let $u \in \mathbb{H}_i \subseteq \mathbb{H}$ be arbitrary. Since $\mathbb{H}_i$ is one-dimensional subspace u is of the form of y in equation 3.4 above. The fact that $\langle Bx, x\rangle$ turned out to be homogeneous and real, we have that $\langle Bu, u\rangle \in \mathbb{R}$ for all $u \in \mathbb{H}_i$. The choice of $\mathbb{H}_i$ was arbitrary, consequently, the result is true for all nonzero subspaces $\mathbb{H}_i$ as well as all $u \in \mathbb{H}$.

Therefore, $\langle Bx, x\rangle \in \mathbb{R}$ for all $u \in \mathbb{H}$.

Corollary 3.1

If B is a commutant of the unbounded Self-adjoint operators with simple spectra F, then

i. B is Self-adjoint
ii. $\sigma(B) \in \mathbb{R}$.

Proof

(i.) Let $x \in \mathbb{H}$ where $\mathbb{H}$ is a Complex Hilbert space and $\|x\| = 1$. From proposition 3.1, $\langle Bx, x\rangle \in \mathbb{R}$, hence

$$\langle Bx, x\rangle = \overline{\langle Bx, x\rangle} = \langle x, Bx\rangle = \langle B^*x, x\rangle$$

Hence $\langle Bx, x\rangle - \langle B^*x, x\rangle = 0$ implying that

$$\langle Bx - B^*x, x\rangle = \langle (B - B^*)x, x\rangle = 0.$$

The operator B is bounded then by Cauchy Swartz inequality

$$|\langle (B - B^*)x, x\rangle| \leq \|\langle (B - B^*)x\rangle\| \, \|x\| = \|(B - B^*)x\|$$

When $\langle (B - B^*)x, x\rangle = 0$ we have $\|(B - B^*)x\| = 0$. Since $x \neq 0$ and $\mathbb{H}$ is a complex Hilbert space, we must have $B - B^* = 0$ or $B = B^*$. Hence B is a Self-adjoint operator.

(ii). Let $x \in \mathbb{H}$ for $x \neq 0$ and $\lambda \in \sigma(B)$, then $(B - \lambda I)x = 0$. Hence

$$0 = \langle (B - \lambda I)x, x\rangle = \langle Bx, x\rangle - \langle \lambda x, x\rangle = \langle Bx, x\rangle - \lambda\langle x, x\rangle = \langle Bx, x\rangle - \lambda\|x\|^2$$

Consequently, $\langle Bx, x\rangle - \lambda\|x\|^2 = 0$ or $\lambda = \frac{\langle Bx,x\rangle}{\|x\|^2}$ for all $x \in \mathbb{H}$. $\lambda = \frac{\langle Bx,x\rangle}{\|x\|^2} \in \mathbb{R}$ since $\langle Bx, x\rangle \in \mathbb{R}$. This is true for all $\lambda \in \sigma(B)$., then $\sigma(B) \in \mathbb{R}$.

Corollary 3.2

Let B be a commutant of the unbounded Self-adjoint operator, F, with simple spectra having the spectral measure $P(\lambda_i)$, $\lambda_i \in \sigma(F|_{\mathbb{H}_i})$. If $\alpha \in \sigma(B)$, then $\alpha = C_p\langle Bx, P(\lambda)x\rangle$ for some positive constant C_p.

Proof

Let $\mathbb{H}_i$ be a nonzero arbitrary subspace of $\mathbb{H}$ such that $\mathbb{H} = \bigoplus_{i\in\mathbb{N}} \mathbb{H}_i$. From the proof of proposition 3.1 above, $\text{Span}(P(\lambda_i)x) = \mathbb{H}_I$ hence for an eigenvector $y \in \mathbb{H}_i$ of $B|_{\mathbb{H}_i}$ and $\|x\| = 1$, there is

$\gamma \in \mathbb{F}$, $\gamma \neq 0$ such that

$$y = \gamma\, P(\lambda)x$$

By the commutativity relation, $BP(\lambda) = P(\lambda)B$, we have

$$\begin{aligned}\langle By, y\rangle &= \langle B\gamma P(\lambda)x, \gamma P(\lambda)x\rangle \\ &= |\gamma|^2\langle BP(\lambda)x, P(\lambda)x\rangle \\ &= \gamma^2\langle P(\lambda)Bx, P(\lambda)x\rangle \\ &= \gamma^2\langle Bx, P^2(\lambda)x\rangle \\ &= \gamma^2\langle Bx, P(\lambda)x\rangle.\end{aligned}$$

We also have that

$$\langle By, y\rangle = \langle \alpha_i y, y\rangle = \alpha_i\|y\|^2 = \alpha_i$$

where α_i in an eigenvalue of $B|_{\mathbb{H}_i}$ corresponding to an eigenvector y. Thus, $\alpha_i = \langle By, y\rangle = \gamma^2\langle Bx, P(\lambda)x\rangle$. The value $\alpha_i \in \sigma\big(B|_{\mathbb{H}_i}\big) \subseteq \sigma(B)$, further, $\mathbb{H}_i$ was arbitrary, hence α_i represents any $\alpha \in \sigma(B)$. Taking $C_p = \gamma^2 > 0$, we get

$$\alpha = C_p\langle Bx, P(\lambda)x\rangle.$$

From $y = \gamma P(\lambda)x$, we have $1 = \|y\| = |\gamma|\ \|P(\lambda)x\|$ as thus,

$$C_p = \gamma^2 = \frac{1}{\|P(\lambda)x\|^2}$$

Proposition 3.2

The spectral measure of the unbounded Self-adjoint operator with simple spectra is a scalar multiple of that of its commutant.

Proof

From equation 3.1 and 3.2 the complex Hilbert space $\mathbb{H} = \bigoplus_{i\in\mathbb{N}} \mathbb{H}_i$ and $\text{Span}(P(\lambda_i)x) = \mathbb{H}_I$ where x is a cyclic vector of $F|_{\mathbb{H}_I}$.

Let $\mathfrak{T} \subseteq \mathbb{R}$ be a closed compact such that $\sigma(B) \in \mathfrak{T}$ where B is a commutant of the unbounded Self-adjoint operator with simple spectra, F. By corollary 3.1, B is a Self-adjoint operator and by definition of commutativity provided, it is bounded. Then by the bounded version of the spectral theorem for the Self-adjoint operator, there exists a unique spectral measure on the Borel sigma-algebra of $\mathfrak{T}$ such that

$$B = \int_{\mathfrak{T}} \lambda_i^* dP(\lambda_i^*)$$

where $\lambda_i^* \in \sigma(B)$, [1,2]. Since the underlying space is a separable Hilbert space, let $P(\lambda_i^*)$ be a spectral projection on one of the closed subsets whose orthogonal direct sum is $\mathbb{H}$. To be precise, let it be a spectral projection on $\mathbb{H}_i$. Since $x \in \mathbb{H}_i$ by definition, then $P_B(\lambda_i^*)x \in \mathbb{H}_i$.

The subspace $\mathbb{H}_i$ is span by $(P_F(\lambda_i)x$ then there is a nonzero constant a such that

$$P_B(\lambda_i^*)x = a(P_F(\lambda_i)x.$$

Consequently,

$$P_B(\lambda_i^*)x - aP_F(\lambda_i)x = \mathbf{0}$$

$$\left(P_B(\lambda_i^*) - aP_F(\lambda_i)\right)x = \mathbf{0}$$

$$\left\|\left(P_B(\lambda_i^*) - aP_F(\lambda_i)\right)x\right\| = 0$$

Since our subspace is a Complex Hilbert space, the $\left\|\left(P_B(\lambda_i^*) - aP_F(\lambda_i)\right)x\right\| = 0$ implies that $P_B(\lambda_i^*) - aP_F(\lambda_i) = 0$, equivalently,

$$P_B(\lambda_i^*) = aP_F(\lambda_i) \quad or \qquad P_F(\lambda_i) = bP_B(\lambda_i^*)$$

where $b = \frac{1}{a}$. This is valid since $P_F(\lambda_i)$ is almost everywhere finite. The values λ_i and λ_i^* are such that

$\lambda_i \in \sigma\left(F|_{\mathbb{H}_i}\right) \subseteq \sigma(F)$ and $\lambda_i^* \in \sigma\left(B|_{\mathbb{H}_i}\right) \subseteq \sigma(B)$ implying that $\lambda_i \in \sigma(F)$ and $\lambda_i^* \in \sigma(B)$, therefore, the above result is valid for any $\lambda \in \sigma(F)$ and $\lambda^* \in \sigma(B)$, hence

$$P_F(\lambda) = bP_B(\lambda^*).$$

REFERENCES

[1]. Birman, M. S. & Solomjak, M.Z. *Spectral theory of Self-adjoint operators in Hilbert space*. 1996. D. Reidel Publishing Company: Holland.

[2]. Gohberg, I. Goldberg, S. & Kaashoek, A. M. *Classes of Linear Operators Vol 1*. 1990. Springer Basel AG; Basel, Switzerland.

[3]. Griffel, D. H. *Applied Functional analysis*. 1985. Ellis Horwood; New York.

[4]. Reed, M. & Simon, B. *Methods of Modern Mathematical Physics I. Functional analysis.* 1972. Academic Press: New York.

[5].Schmüdgen, K. *Unbounded Self-adjoint Operators on Hilbert Space*. 2012. New York: Springer GTM 265.